

UNIFORMIZATION

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The main result of this note is the following uniformization theorem.

Theorem. *Let X be a connected nonsingular complex projective variety of dimension n without elliptic curves, with ample canonical bundle, and very large residually finite fundamental group. Then its universal covering \tilde{X} is a bounded holomorphically convex domain in \mathbf{C}^n .*

The uniformization problem for higher-dimensional varieties was proposed by Weierstrass [W, pp. 95, 232, 304] and Hilbert (22nd problem). The above theorem can be viewed as a converse to the Poincaré ampleness theorem [K, Theorem 5.22].

We will describe the idea of the proof. We take an appropriate Zariski open subset $Z \subset X$ whose universal covering is a bounded domain $\tilde{Z} \subset \mathbf{C}^n$. We construct a one-to-one holomorphic map from the inverse image of Z on \tilde{X} into \tilde{Z} . Finally, we extend the latter inclusion to a holomorphic inclusion $\tilde{X} \hookrightarrow \mathbf{C}^n$.

1. PRELIMINARIES

1.1. Definitions and assumptions. Let $\mathbf{P}^r = \text{Proj } \mathbf{C}[z_0, z_1, \dots, z_r]$ be a projective space. Unless stated otherwise, we assume $X \subset \mathbf{P}^r$ is a connected nonsingular complex projective variety of dimension n . We denote by \tilde{X} its universal covering. The fundamental group $\pi_1(X)$ is called *large* if \tilde{X} contains no proper holomorphic subsets of strictly positive dimension or, what is the same according to Kollár, for every normal cycle $w : W \rightarrow X$, the group $\text{im}[\pi_1(W) \rightarrow \pi_1(X)]$ is infinite [K].

Unless stated otherwise, we also assume X has no elliptic curves, its canonical bundle, denoted by \mathcal{K}_X , is ample, and $\pi_1(X)$ is residually finite and large. The definition of *very large* $\pi_1(X)$ is given in (2.2.2).

A holomorphic space is called *holomorphically convex* if for any given infinite discrete subset D , there is a holomorphic function on the space which is unbounded on D . It is well known that a holomorphic space is *Stein* if and only if it is holomorphically convex and contains no proper holomorphic subsets of strictly positive dimension.

1.2. Griffiths' theorem. One of the main ingredients in the proof of our theorem is the well-known theorem of Griffiths to the effect that given a point Q on a nonsingular quasiprojective variety $X \subset \mathbf{P}^r$, we may choose a Zariski open neighborhood of Q whose universal covering is a bounded domain in \mathbf{C}^n [G].

Further, it follows from [G] that one can choose a finite affine Zariski covering of X , denoted by $\{Z_1, \dots, Z_u\}$, such that the universal covering \tilde{Z}_i of Z_i is a bounded domain and all $\pi_1(Z_i)$ are isomorphic residually finite groups without torsion and have the same closed, normal, residually finite subgroups (identified under the isomorphism). This follows from the Griffiths theorem since we have a great choice of the neighborhoods while each $\pi_1(Z_i)$ is a finitely generated group. Furthermore, we can assume that each Z_i is a complement of a hyperplane section of X (by taking a Veronese map if necessary).

1.3. Fubini spaces. The space \mathbf{C}^{r+1} , with coordinates z_0, z_1, \dots, z_r , will be considered as a standard Euclidean space with a norm $\sqrt{|z_0|^2 + \dots + |z_r|^2}$. Given

a Hilbert space H , we denote by H^* its dual and by $\mathbf{P}(H^*)$ the corresponding projective space with the standard Fubini-Study metric [C, Chap. 4].

For every i , we will fix a countable cofinal set $\{Z_{i,\gamma}\}_{\gamma \in \mathbf{N}}$ of finite Galois coverings of Z_i with the Galois groups independent of i ($1 \leq i \leq u$). By the Riemann existence theorem, each covering $Z_{i,\gamma}$ is naturally compactified to a normal projective variety $M_{i,\gamma}$ and a finite morphism $\phi_{i,\gamma} : M_{i,\gamma} \rightarrow X$. We assume the embedding

$$Z_i \subset X \hookrightarrow \mathbf{P}^r$$

is given by a very ample sheaf $\mathcal{O}_X(1) = \mathcal{K}_X^{\otimes m}$ ($m \gg 0$). The *metric* and *measure* on X and Z_i are determined uniquely by this embedding. Here we consider \mathbf{P}^r with the standard Fubini-Study metric given by the potential

$$\log(1 + \sum_{t=1}^r |\zeta_t(x)|^2), \quad x \in X,$$

in the affine coordinate system $\zeta_1 = z_1/z_0, \dots, \zeta_r = z_r/z_0$. The volume form on X can be written locally $dv = \rho \cdot (\sqrt{-1})^n dx_1 \wedge d\bar{x}_1 \cdots \wedge dx_n \wedge d\bar{x}_n$, where $\rho = \rho_X(x)$ is a positive function and x_1, \dots, x_n are local coordinates on X . On finite Galois coverings, it induces volume forms and we employ the same notation, ρ and dv .

1.4. Positive reproducing kernels and Bergman pseudometrics. Let U denote an arbitrary complex manifold. Let $B(z, w)$ be a Hermitian positive definite complex-valued function on $U \times U$ which means:

- (i) $\overline{B(z, w)} = B(w, z), \quad B(z, z) \geq 0;$
- (ii) $\forall z_1, \dots, z_N \in U, \quad \forall a_1, \dots, a_N \in \mathbf{C} \implies \sum_{j,k}^N B(z_k, z_j) a_j \bar{a}_k \geq 0.$

If $B(z, w)$ is, in addition, holomorphic in the first variable then B is the reproducing kernel of a *unique Hilbert space* H of holomorphic functions on U (see Aronszajn [A] and the articles by Faraut and Korányi in [FK, pp.5-14, pp.187-191]). The evaluation at a point $Q \in U$, $e_Q : f \mapsto f(Q)$, is a continuous linear functional on H .

Conversely, given a Hilbert space H of holomorphic functions on U with all evaluation maps continuous linear functionals then, by the Riesz representation theorem, for every $w \in U$ there exists a unique function $B_w \in H$ such that $f(w) = (f, B_w)$ ($\forall f \in H$) and $B(z, w) := B_w(z)$ is the reproducing kernel for H (which is Hermitian positive definite).

If we assume, *in addition*, that $B(z, z) > 0$ for every z , then we can define $\log B(z, z)$ and a positive semidefinite Hermitian form, called the Bergman pseudo-metric

$$ds_U^2 = 2 \sum g_{jk} dz_j d\bar{z}_k, \quad g_{jk} = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log B(z, z).$$

We get a natural map

$$\Upsilon : U \longrightarrow \mathbf{P}(H^*).$$

In Section 2.2, the function $B(z, w)$ is replace by a section of a relevant bundle.

2. REPRODUCING KERNELS, METRICS, AND HOLOMORPHIC CONVEXITY

2.1. Reproducing kernels for Riemann surfaces. First, we will consider some results of Kazhdan (unpublished), Rhodes [R], and Jorgenson and Kramer [JK].

Let $C := \Gamma \backslash \mathbf{H}$ denote a compact Riemann surface of genus $g(C) \geq 2$, where \mathbf{H} is the upper half-plane and Γ a discrete subgroup of $PSL_2(\mathbf{R})$. Let $\{C_\alpha\}$ be a cofinal system of finite Galois coverings of C . Set $\Gamma_\alpha := \pi_1(C_\alpha)$. Let ℓ be a sufficiently large even integer. One can estimate the following function on \mathbf{H} :

$$(1) \quad \mathfrak{F}_{\ell/2, \alpha}(z, \bar{z}) := {}^{\ell/2}\sqrt{(y^{\ell/2})^2} \cdot \left(\sum_{j=1}^{n_\alpha} |f_j|^2 \right)^{2/\ell} = y^2 \cdot \left(\sum_{j=1}^{n_\alpha} |f_j|^2 \right)^{2/\ell},$$

where $\{f_1, \dots, f_{n_\alpha}\} \subset S_\ell(\Gamma_\alpha)$ is an orthonormal basis of cuspidal forms and $y = \text{Im}(z)$. For $\beta = \alpha + 1$, the basis of $S_\ell(\Gamma_\beta)$ is formed by adding forms to the basis of $S_\ell(\Gamma_\alpha)$. In [JK], the authors considered only cuspidal forms of weight 1 (in our notation), denoted by $S_2(\Gamma_\alpha)$, but the argument works for $S_\ell(\Gamma_\alpha)$ if we replace the heat kernel $K_\alpha^{(1)}(t, z, w)$ by $K_\alpha^{(\ell/2)}(t, z, w)$. The upshot is that

$$B_{\mathbf{H}, \ell}(z, w) := \lim_{\alpha} \sum_{j=1}^{n_\alpha} f_j(z) \overline{f_j(w)} \quad (z, w \in \mathbf{H})$$

is a holomorphic function in z and antiholomorphic function in w .

2.2. Reproducing kernel for \tilde{X} . We assume that X satisfies the assumptions of Section 1.1. Let X_γ be a cofinal system of finite Galois coverings of X . Set $U := \tilde{X}$. We may consider a natural $\pi_1(X)$ -invariant Hermitian metric $h(\cdot, \cdot)$ on $\mathcal{K}_U^{\otimes m}$, as in [K, Chap. 5.13, 7.1, 5.12]. By [K, Chap. 5.13], the natural Hermitian metric on K_U is equivalent to any other $\pi_1(X)$ -invariant Hermitian metric on \mathcal{K}_U . In Section 1.4, we may replace complex-valued functions by sections of relevant bundles.

We consider the bundles $\mathcal{E}_{\gamma, m} := \mathcal{K}_{X_\gamma}^{\otimes m}$ and $\mathcal{E}_{U, m} := \mathcal{K}_U^{\otimes m}$ for an appropriate fixed m so that each $\mathcal{E}_{\gamma, m}$ is very ample; such an m exists [K, Chap. 16.5]. They are equipped with the natural Hermitian metrics $h(\cdot, \cdot)$ and $\|f(z)\| = \sqrt{h(f(z), f(z))}$, as in [K, Chap. 5.13, 7.1, 5.12]. For each γ , we consider a vector space of integrable forms of weight m on X_γ with a norm that behave well in the tower of coverings:

$$\frac{1}{\text{vol}(X_\gamma)} \int_{X_\gamma} \|f\|^2 \rho^{-m} dv.$$

The union of these vector spaces is a pre-Hilbert $pH_{X, m}$ space whose completion is denoted by $H_{X, m}$.

Remark 2.2.1. Clearly, each X_γ satisfies the above assumptions. Given a finite number of points on \tilde{X} , we can assume they lie on a connected open Riemann surface $Y \subset \tilde{X}$ which is a Galois covering of a curvilinear section of X_γ ($\gamma \gg 0$, γ depends on the points) by Deligne's theorem [K, Theorem 2.14.1].

We will show that the corresponding $H_{X, m}$ has a reproducing kernel provided $\pi_1(X)$ is very large (see Definition 2.2.2 below). We consider $U \times U$ and write a point of $U \times U$ as a pair (z, w) . The reproducing kernel will be a section of a vector bundle $p_1^* \mathcal{E}_{U, m} \otimes p_2^* \mathcal{E}_{U, m}$ where p_1 and p_2 are the coordinate projections of $U \times U$.

Let $\{u_k\} \subset pH_{X,m}$ be an orthonormal basis constructed similarly to the basis of $\bigcup_{\alpha} S_{\ell}(\Gamma_{\alpha})$. We set

$$B_{U,m}(z, w) := \sum p_1^* u_k(z) \otimes p_2^* \bar{u}_k(w)$$

and verify that the series converges and the sum is a section holomorphic in the first variable. We take a general hyperplane section $\tau : D \hookrightarrow X$ through a point $x \in X$ and consider a standard exact sequences with $D = mK_X$ (m -th power of the canonical divisor):

$$0 \longrightarrow \mathcal{K}_X \longrightarrow \mathcal{K}_X(D) \xrightarrow{P.R.} \mathcal{K}_D \longrightarrow 0$$

where $P.R.$ is the Poincaré residue map, and $\tau^*(\mathcal{K}_X(D)) = \mathcal{K}_D$. We also have a natural inclusion

$$\mathcal{O}_X(mK) = \mathcal{O}_X(D) \hookrightarrow \mathcal{K}_X(D) = \mathcal{O}_X((m+1)K).$$

The natural Hermitian metric on $\mathcal{K}_X(D)$ will induce a Hermitian metric on \mathcal{K}_D as well as on $\mathcal{O}_X(D)$. Furthermore, all metrics behave well in the tower of Galois coverings of X . By trivial induction, we are reduced to the one-dimensional case. Let Y denote the inverse image on \tilde{X} of a general curvilinear section $C \subset X$ through a point of X . It is connected. One can estimate on Y the function similar to the expression (1). It follows that the series converges pointwise hence uniformly by Dini's theorem, and $B_{U,m}(z, w)$ satisfies the condition 1.4(i).

Definition 2.2.2. With the above assumptions and notation, we say $\pi_1(X)$ is *very large* if $B_{U,m}(z, w)$ satisfies the conditions 1.4(i) and 1.4(ii) (sheaf-theoretic version), for all $m \gg 0$.

Let $Q \in \tilde{X}$ be an arbitrary point. Consider a coordinate neighborhood of Q in \tilde{X} . In that neighborhood, the elements of $H_{X,m}$ become functions and we may consider the evaluation map e_Q . With assumptions of the theorem, e_Q is a continuous linear functional. So, for $m \gg 0$, $H_{X,m}$ is a *unique Hilbert space* with reproducing kernel $B_{U,m}(z, w)$ (see Faraut's article in [F-K, pp.5-12]). We obtain a holomorphic *embedding* of U into the corresponding infinite-dimensional projective space by the Schwarz inequality (see Korányi's article in [FK, p.191]). Indeed, each finite covering is embedded in the corresponding finite-dimensional projective space.

Thus, we get a metric on \tilde{X} and a natural isometric and holomorphic embedding

$$\Upsilon : \tilde{X} \hookrightarrow \mathbf{P}(H_{X,m}^*).$$

Proposition 2.3. *With the above notation and assumptions of the theorem, the functional element of the diastasis generates a strictly plurisubharmonic function on \tilde{X} , and \tilde{X} is a Stein manifold.*

Proof. We have two expressions for the diastasis on \tilde{X} [C, (5) and (29)]:

$$(2) \quad D(\mathbf{p}, \mathbf{q}) = \Phi(z(\mathbf{p}), \overline{z(\mathbf{p})}) + \Phi(z(\mathbf{q}), \overline{z(\mathbf{q})}) - \Phi(z(\mathbf{p}), \overline{z(\mathbf{q})}) - \Phi(z(\mathbf{q}), \overline{z(\mathbf{p})}),$$

$$(3) \quad D(\mathbf{p}, \mathbf{q}) = \log \frac{(\sum_{\sigma=0}^{\infty} |\xi^{\sigma}(\mathbf{p})|^2) (\sum_{\sigma=0}^{\infty} |\xi^{\sigma}(\mathbf{q})|^2)}{|\sum_{\sigma=0}^{\infty} \xi^{\sigma}(\mathbf{p}) \overline{\xi^{\sigma}(\mathbf{q})}|^2},$$

where $\Phi(z, \bar{z})$ is a real-valued analytic potential of our metric [C, Chap.2] and $[\xi^0 : \xi^1 : \dots]$ are homogeneous coordinates in the projective space $\mathbf{P}(H_{X,m}^*)$. While (3) is a global expression for the diastasis, (2) is the functional element of the diastasis.

For a fixed \mathbf{p} , from (3), we see that the diastasis on \tilde{X} is single-valued, non-negative, and real analytic in \mathbf{q} , except at the intersection of \tilde{X} in $\mathbf{P}(H_{X,m}^*)$ with the antipolar hyperplane of \mathbf{p} where $D(\mathbf{p}, \mathbf{q})$ becomes infinite, i.e., the denominator in (3) vanishes. This intersection is clearly either empty or a codimension one subvariety in \tilde{X} [C, pp. 19-20, Theorems 11, 12 and Corollaries 1, 3].

In (2), we observe that the functional element of the diastasis generates a single-valued function on \tilde{X} . Indeed, it generates a function on Y , the preimage on \tilde{X} of a general curvilinear section through a finite number of points of X_γ ($\gamma \gg 0$, γ depends on the chosen points). The metric and diastasis on Y are induced from \tilde{X} [C, Chap.2]. Here we use that every holomorphic bundle on a noncompact Riemann surface is trivial, and we can assume that Y is noncompact and connected by Deligne's theorem [K, Theorem 2.14.1] since $\pi_1(X)$ is large. The observation, now, follows since \tilde{X} is simply connected.

Thus, the functional element of the diastasis generates a strictly plurisubharmonic *function* on \tilde{X} . Since X is compact, the proposition, now, follows from the Oka-Grauert-Narasimhan solution of the Levi problem.

3. COMPARISON OF HILBERT SPACES

The aim of this section is to construct a finite-dimensional linear system on $Z'_{i,\gamma}$ (see 3.1) that together with a pullback of a linear system on \tilde{X} will give an embedding of $Z'_{i,\gamma}$ into projective space. Furthermore, such linear systems will be closely related for different i 's ($1 \leq i \leq u$); see Proposition 3.9 below.

3.1. Notation and assumptions. We keep the notation of Section 1 and assumptions of the theorem. Furthermore, we assume $\phi_{i,\gamma}$ can not be extended to an étale covering of X . We get the following diagram of base extensions:

$$\begin{array}{ccccc} Z'_{i,\gamma} & \xrightarrow{\phi'_{i,\gamma}} & Z_{i,\tilde{X}} & \longrightarrow & \tilde{X} \\ \downarrow \xi_{i,\gamma} & & \downarrow \phi_i & & \downarrow \beta \\ Z_{i,\gamma} & \xrightarrow{\phi_{i,\gamma}} & Z_i & \longrightarrow & X. \end{array}$$

Let $s \in \mathbf{N}$ denote a sufficiently large integer. We consider an arbitrary square integrable, or L^2 -integrable, section that is a restriction of a section over $M_{i,\gamma}$:

$$\omega_{i,\gamma,s} \in H^0(Z_{i,\gamma}, \mathfrak{K}_{M_{i,\gamma}}^{\otimes s}), \quad \|\omega_{i,\gamma,s}\|^2 = \frac{1}{\text{vol}(Z_{i,\gamma})} \int_{Z_{i,\gamma}} |\omega_{i,\gamma,s}|^2 \rho^{-s} dv < \infty.$$

Here $\omega_{i,\gamma,s}$ is viewed as an automorphic form of weight s on \tilde{Z}_i , and ρ arises from ρ_X . The pullback gives a section $\omega'_{i,\gamma,s} \in H^0(Z'_{i,\gamma}, \xi_{i,\gamma}^*(\mathfrak{K}_{M_{i,\gamma}}^{\otimes s}|_{Z_{i,\gamma}}))$.

We denote by $\mathcal{A}_{\tilde{Z}_i}$ the space of holomorphic functions on \tilde{Z}_i , and by B the corresponding reproducing kernel. To simplify notation, we specify only the first rows of the matrices in Lemma 3.4 below.

3.2. Compact exhaustions. Let $K_{i,1} \subset \cdots \subset K_{i,t} \subset \cdots \subset Z_{i,\tilde{X}} = Z_i \times_X \tilde{X}$ be a compact exhaustion of $Z_{i,\tilde{X}}$ such that

$$\tilde{K}_1 \subset \cdots \subset \tilde{K}_t \subset \cdots \subset \tilde{X} \quad (\tilde{K}_t = \bigcup_{i=1}^u K_{i,t})$$

is a compact exhaustion of X . We assume each compact is a union of compact neighborhoods. Let $\Phi_i \subset \tilde{Z}_i$ denote the fundamental region of $Z'_{i,\gamma}$, and $K_{i,t,\Phi_i} \subset \Phi_i$ denote the inverse image of $K_{i,t}$. For a given t , we can find a sufficiently high finite Galois covering $\beta_\mu : X^\mu \rightarrow X$ such that \tilde{K}_t projects one-to-one into X^μ while $K_{i,t}$ projects into $Z_{i,X^\mu} := \beta_\mu^{-1}(Z_i)$. Similarly, one defines Φ_i^μ and $K_{i,t,\Phi_i^\mu} \subset \Phi_i^\mu$. Set

$$Z_{i,\gamma}^\mu := Z_{i,\gamma} \times_{Z_i} Z_{i,X^\mu}, \quad \Sigma^\mu := \text{Gal}(\tilde{Z}_i/Z_{i,\gamma}^\mu) \text{ and } \Sigma' := \text{Gal}(\tilde{Z}_i/Z'_{i,\gamma}).$$

Given an index i ($1 \leq i \leq u$), an arbitrary $\epsilon > 0$, and $K_{i,t}$ as above, we can ϵ -approximate the Poincaré series $P_{\omega'_{i,\gamma,s}}$ on $K_{i,t}$ by a global section of $\mathfrak{K}_{\tilde{X}}^{\otimes s}$ since \tilde{X} is a Stein manifold; indeed, $P_{\omega'_{i,\gamma,s}}$ may be viewed as a section of $H^0(K_{i,t}, \mathfrak{K}_{\tilde{X}}^{\otimes s})$.

3.3. System of equations. Let $x \in \tilde{Z}_i \subset \mathbb{C}^n$ be an arbitrary point. Let

$$\Gamma := Gal(Z_{i,\gamma}^\mu/Z_{i,X^\mu}) = \{g_1\Sigma^\mu, g_2\Sigma^\mu, \dots, g_q\Sigma^\mu\}$$

denote the corresponding Galois group with unit element g_1 in $Gal(\tilde{Z}_i/Z_i)$.

For a *fixed* index i , let $J_\lambda(x)$ denote the Jacobian of an automorphism λ on \tilde{Z}_i . Let $m = m(\gamma) \in \mathbf{N}$ denote a sufficiently large integer.

We consider the following system of q equations in q unknowns Y_1, \dots, Y_q at x :

[illegible]

where each $G_v(x)$ ($1 \leq v \leq q$) is an arbitrary $\text{Gal}(\tilde{Z}_i/Z_{i,X^\mu})$ -automorphic form of weight m on \tilde{Z}_i , i.e., $J_\alpha^m(x)G_v(\alpha x) = G_v(x)$ for $\alpha \in \text{Gal}(\tilde{Z}_i/Z_{i,X^\mu})$.

Lemma 3.4 (lifting). *For almost every choice of representatives g_1, \dots, g_q of the group Γ (possibly with a finite number of exceptions) and every $x \in K_{i,t,\Phi_i^*}$,*

- (i) the system (4) has a solution obtained by Cramér's rule, and
- (ii) suitable solutions Y_v 's are functions of $x \in K_{i,t,\Phi_i^\mu}$ and $Y_v(x) = Y_1(g_v x)$ for $v \geq 2$, where $\{Y_v(\cdot)\}$ denotes a solution of (4) at the corresponding point.

Proof. The $D(x) := \text{Det}(J_{g_1}^m(x), \dots, J_{g_q}^m(x))$ is a Vandermonde determinant hence

$$D(x) = J_{g_1}^m(x) \cdots J_{g_q}^m(x) \cdot \prod_{j>k} (J_{g_j}^m(x) - J_{g_k}^m(x)) \quad (J_{g_v g_k^{-1} g_k}(x) \equiv J_{g_v g_k^{-1}}(g_k x) J_{g_k}(x)).$$

Clearly Σ^μ is infinite. Given the unit g_1 and the point x as above, we can choose, in infinitely many ways, new representatives g_v 's in the corresponding cosets such

that $J_{g_v g_1^{-1}}(g_1 x) \neq 1$ ($2 \leq v \leq q$ and $\forall x \in K_{i,t,\Phi_i^\mu}$). The first assertion follows from the uniform convergence of Poincaré series by trivial induction.

To prove (ii), we will show that $Y_k(x) = Y_1(g_k x)$ for $2 \leq k \leq q$ and suitable solutions. We consider a linear system like (4) with x replaced by $g_k x$ in (4) and the determinant $D(g_k x)$. From the identity $J_{g_j g_k}(x) \equiv J_{g_j}(g_k x) J_{g_k}(x)$, we obtain

$$D(g_k x) = \text{Det}(J_{g_1 g_k}^m(x) J_{g_k}^m(x)^{-1}, \dots, J_{g_q g_k}^m(x) J_{g_k}^m(x)^{-1})$$

with $J_{g_1 g_k}(x) J_{g_k}(x)^{-1} \equiv 1$. Clearly $D(g_k x)$ is the determinant of the system with the right-hand side $\{G_v(g_k x)\}$ ($1 \leq v \leq q$). Hence

$$D'(x) := \text{Det}(J_{g_1 g_k}^m(x), \dots, J_{g_q g_k}^m(x))$$

is the determinant of a system with the right-hand side $\{G_v(g_k x) J_{g_k}^m(x)\}$. But $D'(x)$ is also the determinant of the system, similar to (4), in the q unknowns Y_k, \dots, Y_1, \dots , with right-hand side $\{G_v(x)\}$. Clearly $G_v(x) = G_v(g_k x) J_{g_k}^m(x)$ for $1 \leq v \leq q$. It follows that $Y_k(x) = Y_1(g_k x)$ for suitable solutions.

Lemma 3.5. *We fix i and the compact $K_{i,t} \subset X^\mu$ (see 3.2). With holomorphic on \tilde{Z}_i automorphic forms $G_1(x), \dots, G_q(x)$ (see 3.3), let $F(x)$ denote a suitable solution of (4) on the compact $K := K_{i,t,\Phi_i^\mu}$ obtained in Lemma 3.4. Set $F(x) = 0$ on $\tilde{Z}_i \setminus K$. Let $L_m^2(\tilde{Z}_i, \Sigma^\mu, dv) := L_m^2(\tilde{Z}_i, \Sigma^\mu, \rho_X, dv)$ denote the standard Hilbert space (with reproducing kernel). Then*

- (i) $P_{F,K}(x) := \sum_{\sigma \in \Sigma^\mu} F(\sigma x) J_\sigma^m(x) \in L_m^2(\tilde{Z}_i, \Sigma^\mu, dv)$;
- (ii) *we obtain a holomorphic on \tilde{Z}_i automorphic form of weight m with respect to Σ^μ by applying the Bergman projection*

$$(\beta_m P_{F,K})(x) := \int_{\Phi_i^\mu} B_m(x, \xi) P_{F,K}(\xi) \rho^{-m}(\xi) dv_\xi \in L_m^2(\tilde{Z}_i, \Sigma^\mu, dv) \cap \mathcal{A}_{\tilde{Z}_i};$$

- (iii) $P_{F,K}(x) = F(x)$ on K , and $(\beta_m P_{F,K})(x)$ is uniquely determined.

Proof. Since $F(x)$ is bounded, the series in (i) converges absolutely and uniformly, and $P_{F,K}(x)$ is a measurable form in $L_m^2(\tilde{Z}_i, \Sigma^\mu, dv)$. Finally, the uniqueness follows from Cramér's rule and the well-known definition of relative Poincaré series. Recall that the latter definition is independent of the choice of representatives.

3.6. For a fixed i ($1 \leq i \leq u$), we consider a finite basis of extendable sections

$$\Lambda_{i,1}, \dots, \Lambda_{i,\ell} \in H^0(Z_{i,\gamma}, \mathcal{K}_{M_{i,\gamma}}^{\otimes m})$$

that defines an embedding of $M_{i,\gamma}$ and $Z_{i,\gamma}$ into projective space. We can assume that $m = m(\gamma) \gg 0$ and those sections are L^2 -integrable. By Lemmas 3.4 and 3.5, we can find a linearly independent subset of

$$H^0(Z'_{i,\gamma}, \xi_{i,\gamma}^*(\mathcal{K}_{M_{i,\gamma}}^{\otimes m} | Z_{i,\gamma}))$$

that defines an embedding into projective space, consisting of pullbacks of the orthonormal basis of the Hilbert space $H_{X,m}$ supplemented by the pullback of the maximal (finite) number of sections

$$\Omega_{i,1}, \dots, \Omega_{i,p}, \dots \in \{\Lambda_{i,1}, \dots, \Lambda_{i,\ell}\}$$

that are not arising from $H_{X,m}$, and, if possible, a finite number of *new* L^2 -integrable sections. The latter sections are coming from Z_j 's for $j \neq i$; see (3.7) below. For each i , those sections which are *not* arising from $H_{X,m}$ generate a finite-dimensional Hilbert space, denoted by $V_{i,\gamma}$, and $\dim V_{i,\gamma}$ is independent of i . For $1 \leq k \leq u$, let $L_m(\tilde{Z}_k, \Sigma', dv)$ denote the completion of $\bigcup_\mu L_m^2(\tilde{Z}_k, \Sigma^\mu, dv)$.

We consider a sequence $\{\epsilon_t\}$ of small positive real numbers with $\lim \epsilon_t = 0$. Let

$$\begin{array}{ccccc} \mathcal{Z}_i = M_{i,\gamma} \times_X \tilde{X} & \longleftarrow & M_{i,\gamma} \times_X \tilde{X} \times_X M_{j,\gamma} & \longrightarrow & M_{j,\gamma} \times_X \tilde{X} = \mathcal{Z}_j \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Z}_i^t = M_{i,\gamma} \times_X X^t & \xleftarrow{f_i^t} & M_{i,\gamma} \times_X X^t \times_X M_{j,\gamma} & \xrightarrow{f_j^t} & M_{j,\gamma} \times_X X^t = \mathcal{Z}_j^t \end{array}$$

be a natural diagram, where $1 \leq i, j \leq u$ and $X^t \rightarrow X$ is a suitable finite Galois covering, i.e., the compact \tilde{K}_t maps one-to-one onto a compact $K_t \subset X^t$. The maps f_i^t and f_j^t are finite and generically etale.

Set $\Omega_i := \Omega_{i,1}$. Precisely, we have the following correspondence.

3.7. Approximation. Given the element Ω_i , now we will produce an element $\Omega_j \in L_m(\tilde{Z}_j, \Sigma', dv)$ for $1 \leq j \leq u$. By abuse of notation, we denote by Ω_i the pullbacks of Ω_i to \mathcal{Z}_i and \mathcal{Z}_i^t . Our compact exhaustions (see (3.2)) give compact exhaustions of \mathcal{Z}_i . For a suitable t , the corresponding compact set descends to \mathcal{Z}_i^t . With the above notation, let $P_{\Omega_i}(x)$ denote the corresponding Poincaré series (see 3.2). The Poincaré series can be ϵ_t -approximated on $K_{i,t}$ by global sections of $\mathcal{K}_X^{\otimes m}$. We also consider $P_{\Omega_i h_i^w}(x)$, where h_i is the linear form defining $X \setminus Z_i$. Let

$$G_{t,1} h_i^0 \in H^0(\tilde{X}, \mathcal{K}_X^{\otimes m}), \dots, G_{t,q} h_i^{q-1} \in H^0(\tilde{X}, \mathcal{K}_X^{\otimes m+q-1})$$

be the corresponding ϵ_t -approximating sections on the compact $K_{i,t}$, where each $G_{t,v} \in H^0(\tilde{X}, \mathcal{K}_X^{\otimes m})$. By lemmas, we obtain automorphic forms

$$\omega_{j,t} \in L_m^2(\tilde{Z}_j, \Sigma^t, dv) \bigcap \mathcal{A}_{\tilde{Z}_j}, \quad \lim_{t \rightarrow \infty} \omega_{j,t} = \Omega_j \in \mathcal{A}_{\tilde{Z}_j},$$

where the convergence is in the Hilbert space $L_m(\tilde{Z}_j, \Sigma', dv)$. The convergence follows from the assumptions on Ω_i , the uniqueness (see (3.4)-(3.5)), and the diagram in (3.6) that allows to compare \mathcal{Z}_i^t and \mathcal{Z}_j^t . Thus Ω_j is the desired element.

3.8. Notation. Let $H_{i,\gamma,m} := V_{i,\gamma} \oplus H_{X,m}$ denote a Hilbert space, and $A_{\gamma,m}$ the corresponding infinite-dimensional complex linear space (we have applied a forgetful functor). Let $\mathbf{P}(A_{\gamma,m}^*)$ denote the corresponding projective space. We will consider only the linear transformations of $\mathbf{P}(A_{\gamma,m}^*)$ that leave $\mathbf{P}(H_{X,m}^*)$ fixed.

We have proved the following

Proposition 3.9. *We keep the above notation and assumptions of the theorem, and fix γ and $m = m(\gamma) \gg 0$. For all i 's ($1 \leq i \leq u$), there is a natural one-to-one linear correspondence between the Hilbert spaces $H_{i,\gamma,m}$ that induces an isomorphism on the Hilbert subspace $H_{X,m} \subset H_{i,\gamma,m}$. Moreover, $\dim V_{i,\gamma}$ is finite and independent of i .*

4. MONODROMY

4.1. Notation and Assumptions. We keep the notation of Section 1 and assumptions of the theorem. Let $B_i(\tilde{X})$ denote the inverse image of the hyperplane $X \setminus Z_i$ on \tilde{X} . Each $\tilde{X} \setminus B_i(\tilde{X})$ is connected and the natural map $\pi_1(Z_i) \rightarrow \pi_1(X)$ is an epimorphism. We get a natural holomorphic surjection $\phi'_{i,\gamma} : Z'_{i,\gamma} \rightarrow Z_{i,\tilde{X}}$. Set $U_i := Z'_{i,\tilde{X}}$ and $g_i := \phi'_{i,\gamma}$.

We fix an arbitrary point $\tilde{x} \in \tilde{X} \setminus (\bigcup_i B_i(\tilde{X}))$. Let $f(0)$ be a unique natural germ of holomorphic map from \tilde{X} into $U_1 \subset \mathbf{P}(A_{\gamma,m}^*)$ sending \tilde{x} to a point $\tilde{u} \in U_1$ such that $g_1(\tilde{u}) = \tilde{x}$. Finally, we fix the inclusion $\tilde{Z}_1 \subset \mathbf{C}^n$.

Proposition 4.2. *With the above notation and assumptions of the theorem, there exists a unique holomorphic inclusion*

$$f_1 : \tilde{X} \hookrightarrow \tilde{Z}_1 \subset \mathbf{C}^n.$$

In particular, \tilde{X} is holomorphically equivalent to a bounded domain in \mathbf{C}^n .

Proof. We proceed in two steps. First, we will construct a holomorphic map $\Upsilon_\gamma : \tilde{X} \rightarrow \mathbf{P}(A_{\gamma,m}^*)$ and a holomorphic inclusion $\tilde{X} \setminus B_1(\tilde{X}) \hookrightarrow \tilde{Z}_1 \subset \mathbf{C}^n$. Then we will map \tilde{X} into \mathbf{C}^n and verify that we get an inclusion.

Step 1. We take an arbitrary point $P_1 \in \tilde{X}$ and connect it to \tilde{x} by a continuous path $P_t \subset \tilde{X}$ ($0 \leq t \leq 1, P_0 = \tilde{x}$). We will show that there exists one and only one family $f(t)$ ($0 \leq t \leq 1$) of germs of holomorphic maps from \tilde{X} into $\mathbf{P}(A_{\gamma,m}^*)$ sending P_t to $f(t)(P_t)$ such that $f(t)$ induces $f(t^*)$, when P_{t^*} is near enough to P_t , and $f(0)$ is our fixed germ; moreover, $f(t)(P_t) \in U_1$ and $g_1 \cdot f(t)(P_t) = P_t$ provided P_t lies under a point of U_1 . Such a family is called compatible.

We claim that if $f(t)$ ($0 \leq t < 1$) is a compatible family of holomorphic maps which is open on the right side then we can complete it by $f(1)$ to a compatible family $f(t)$ ($0 \leq t \leq 1$).

Let $\phi = \phi^{P_1}$ be a holomorphic map of a small neighborhood $\mathcal{V} \subset \tilde{X}$ of P_1 into $\mathbf{P}(A_{\gamma,m}^*)$ corresponding to a germ sending P_1 to a point in U_i such that $\phi(\mathcal{V}) \subset U_i$ and $g_i \cdot \phi(\mathcal{V}) = \mathcal{V}$. Such a germ always exists.

First, we will assume that $i = 1$, i.e., $P_1 \in \tilde{X} \setminus B_1(\tilde{X})$. Hence P_1 lies under a point $Q \in U_1$ and $\phi(\mathcal{V})$ is a small connected neighborhood in U_1 that projects onto \mathcal{V} , i.e., $g_1 \cdot \phi$ is an identity on \mathcal{V} .

Let P_s be a point of the path which is near enough to P_1 such that $P_t \in \mathcal{V}$ for $s \leq t \leq 1$. Let $\mathcal{W} \subset \mathcal{V}$ be a neighborhood of P_s such that $f(s)$ maps it into $\mathbf{P}(A_{\gamma,m}^*)$. In this case, $f(s) \cdot \phi^{-1}$ gives a holomorphic map from $\phi(\mathcal{W})$ onto $f(s)(\mathcal{W})$.

The key point is that the map $f(s) \cdot \phi^{-1}$ can be *extended uniquely* to a global holomorphic linear map σ of $\mathbf{P}(A_{\gamma,m}^*)$ because $\phi(\mathcal{W})$ spans linearly $\mathbf{P}(A_{\gamma,m}^*)$.

Moreover, σ acts as a deck transformation of U_1 over $\tilde{X} \setminus B_1(\tilde{X})$ because it is a deck transformation over a small neighborhood in $\tilde{X} \setminus B_1(\tilde{X})$. We get $\sigma(U_1) = U_1$ and $g_1 \cdot \sigma(a) = g_1(a)$ for every $a \in U_1$. It is a simple matter to see that the germ $f(1)$ determined by $\sigma \cdot \phi$ at P_1 satisfies our demand.

Now, we will assume that $P_1 \in B_1(\tilde{X})$ and P_1 lies under a point $Q \in U_i$, where $i \neq 1$. A similar argument applies to obtain a local map as well as a global holomorphic linear map σ .

This establishes our claim, i.e., we can complete the family on the right. Because \tilde{X} is simply connected we get a holomorphic map $\Upsilon_\gamma : \tilde{X} \rightarrow \mathbf{P}(A_{\gamma,m}^*)$. Since $\pi_1(Z_i)$ is residually finite, we get a holomorphic inclusion

$$\tilde{X} \setminus B_1(\tilde{X}) \hookrightarrow \tilde{Z}_1 \subset \mathbf{C}^n.$$

Step 2 (inclusion into \mathbf{C}^n). Because $B_1(\tilde{X}) \subset \tilde{X}$ is a holomorphic subset, the above inclusion has a unique holomorphic extension $f_1 : \tilde{X} \rightarrow \mathbf{C}^n$.

We will show that f_1 is a holomorphic inclusion hence \tilde{X} is a bounded domain. If $\dim \tilde{X} = 1$, i.e., $n = 1$ then f_1 is an open bijection onto its image by the structure theorem for holomorphic functions in one variable. We will reduce the general case to the one-dimensional case.

We suppose that f_1 is *not* a one to one map onto its image and derive a contradiction. Let $q \in f_1(\tilde{X}) \subset \mathbf{C}^n$ be an exceptional point, i.e., $f_1^{-1}(q)$ contains more than one point. By Osgood's theorem, there is a point $Q \in f_1^{-1}(q)$ which is *not* an isolated point in $f_1^{-1}(q)$ unless an open neighborhood of q in \mathbf{C}^n is covered exactly k times by an open neighborhood of Q in \tilde{X} . In the latter case, we get $k = 1$, a contradiction. In the former case, we will first assume that $\dim X = 2$.

Case: $\dim X = 2$. Then $f_1^{-1}(q) = B_1(\tilde{X})$ since $B_1(\tilde{X})$ is a nonsingular connected curve. Let $C \subset X$ be a general hyperplane section. Its preimage in \tilde{X} , denoted by $\beta^{-1}(C)$, will intersect $B_1(\tilde{X})$ in an infinite number of points, and an infinite number of points of $\beta^{-1}(C)$ will go to the point q under the map f_1 .

We consider the following two diagrams:

$$\begin{array}{ccc} d \in \Delta & \xleftarrow{f_{1,C}} & \tilde{C} \\ \tau \downarrow & & \downarrow \nu \\ \tilde{u} \in \tilde{Z}_1 & & \tilde{X} \\ \alpha \downarrow & & \downarrow \beta \\ c \in Z_1 & \longrightarrow & X \end{array} \quad \begin{array}{ccc} \Delta & \xleftarrow{f_{1,C}} & \tilde{C} \\ \tau \downarrow & & \downarrow \nu \\ \mathbf{C}^n & \xleftarrow{f_1} & \beta^{-1}(C). \end{array}$$

Here the disk $\Delta \subset \mathbf{C}$ denotes the universal covering of $C \cap Z_1$, as well as of its preimage under α by [K, Theorem 2.14.1], and $\tau(\Delta) \subset \tilde{Z}_1 \subset \mathbf{C}^n$. The existence of the holomorphic inclusion $f_{1,C}$ was just established. We will show that the map f_1 , restricted to $\beta^{-1}(C)$, is also an inclusion *into* \tilde{Z}_1 .

We claim that the second diagram is commutative provided we have fixed a point $c \in C \cap Z_1$ and have chosen a point $\tilde{u} \in \tilde{Z}_1$ over c and a point $d \in \Delta$ over \tilde{u} together with their respective small complex neighborhoods. Indeed, on $\tilde{C} \setminus B_1(\tilde{C})$, we get

$$\tau \cdot f_{1,C} = f_1 \cdot \nu$$

by the constructions of the maps f_1 and $f_{1,C}$ employing analytic continuations along paths, as in Step 1; $B_1(\tilde{C})$ is defined similarly to $B_1(\tilde{X})$. The equality holds on \tilde{C} by continuity. It follows that the point q actually belongs to \tilde{Z}_1 .

Furthermore, τ is clearly an open map from Δ onto $\tau(\Delta)$ hence $\tau \cdot f_{1,C}$ is an open map from \tilde{C} onto $\tau \cdot f_{1,C}(\tilde{C})$. Its image contains a small open disk about the point

q , say $\mathcal{V}_q \subset \tau \cdot f_{1,C}(\tilde{C})$. We have a holomorphic inclusion $f_1^{-1} : \mathcal{V}_q \setminus q \hookrightarrow \beta^{-1}(C)$. We also have a holomorphic map

$$f_2 : \beta^{-1}(C) \longrightarrow \tilde{Z}_2 \subset \mathbf{C}^n$$

where f_2 and \tilde{Z}_2 are similar to f_1 and \tilde{Z}_1 . Clearly we may assume that $f_2 \cdot f_1^{-1}|_{\mathcal{V}_q \setminus q}$ is an inclusion into a bounded domain in a copy of \mathbf{C}^n since C was a *general* hyperplane section. It follows that the map $f_1^{-1}|_{\mathcal{V}_q \setminus q}$ can be extended to the point q . Hence $f_1^{-1}(q)$ can not contain more than one point, a contradiction.

Case: $\dim X \geq 3$. We take a general 2-dimensional linear section $Y \subset X$ that intersects the image of $f_1^{-1}(q)$ in X in at least two distinct points; this is possible since not every point of $f_1^{-1}(q)$ is isolated in $f_1^{-1}(q)$. By the Lefschetz-type theorem, the natural map $\pi_1(Y) \rightarrow \pi_1(X)$ is an isomorphism. Furthermore, Y satisfies all the assumptions of the theorem and $\dim Y = 2$, a contradiction.

This proves the proposition and the theorem.

Remark 4.3 (Campana). The universal covering of a sufficiently general ample divisor X in a simple Abelian threefold is not a bounded domain, since it follows from [LS] that all bounded holomorphic functions on \tilde{X} are constants; $\pi_1(X)$ is large and residually finite, and \mathcal{K}_X is ample.

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